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On an Identity Theorem in the Nevanlinna Class $\mathcal{N}$

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We prove the following theorem: Let f be in the Nevanlinna class $\mathcal{N}$, and let z_n be distinct points in the unit disk D with $\sum_{n=1}^{\infty}(1-|z_n|)=\infty$. Further let $\lambda_n>0$, $\lambda_n\to\infty$ as $n\to\infty$ and $\varepsilon_n>0$, $\sum_{n=1}^{\infty}\varepsilon_n<\infty$.
If

$$|f(z_n)|<\exp\left(-\frac{\lambda_n}{1-|z_n|}-\frac{1}{\delta_n^2}\right), \qquad n=1,2,...,$$

where

$$\delta_n=\min\{\varepsilon_n, \tfrac{1}{2}\inf_{\substack{i\in\mathbb{N}\\ i\neq n}}|z_n-z_i|\}, \qquad n\in\mathbb{N},$$

then $f\equiv 0$. This result is an extension of the classical theorem of Blaschke about the zeros of functions in the Nevanlinna class $\mathcal{N}$, in the case when these zeros are distinct.

1. INTRODUCTION

Let $\{z_n\}$ be a sequence of distinct points in $D=\{z\in\mathbb{C}, |z|<1\}$ with $\sum_{n=1}^{\infty}(1-|z_n|)=\infty$, and let $\mathcal{N}$ denote the Nevanlinna class of analytic functions of bounded characteristic in D. It is well known that $\mathcal{N}$ contains all H^p functions for every p, $0<p\leqslant\infty$ (see [2, p. 16]).

We ask the following question: How quickly can the values of a non-constant function in $\mathcal{N}$ on $\{z_n\}$ approximate an arbitrary number in $\mathbb{C}$?

Equivalently this question can be formulated in the following problem: Given a sequence $\{z_n\}$ as above, describe the sequences $\{a_n\}$, $a_n>0$, $n\in\mathbb{N}$, $a_n\to 0$ as $n\to\infty$, for which there is a function $f\not\equiv 0$ in $\mathcal{N}$ such that

$$|f(z_n)|<a_n \qquad \text{for every} \quad n\in\mathbb{N}.$$

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In this paper we prove that a sequence $\{a_n\}$ cannot satisfy the condition of this problem, if it tends to zero quicker then a certain sequence $\{a_n^*\}$, which we give as a concrete expression of n and of $\{z_n\}$.

Our proposition is the following:

THEOREM. *Let $\{z_n\}$ be a sequence of distinct points in D for which $\sum_{n=1}^{\infty}(1-|z_n|)=\infty$. Further let $\{\lambda_n\}$ and $\{\varepsilon_n\}$ be two sequences of positive numbers, such that $\lambda_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. If a function $f\in\mathcal{N}$ satisfies the inequality*

$$|f(z_n)|<a_n^*=\exp\left(-\frac{\lambda_n}{1-|z_n|}-\frac{1}{\delta_n^2}\right) \qquad \text{for all} \quad n\in\mathbb{N},$$

where

$$\delta_n=\min\{\varepsilon_n, \tfrac{1}{2}\inf_{\substack{i\in\mathbb{N}\\ i\neq n}}|z_n-z_i|\}, \qquad n\in\mathbb{N},$$

then $f\equiv 0$.

This means that the only sequences $\{z_n\}$ of distinct points in D, for which the inequality $|f(z_n)|<a_n^*$, $n\in\mathbb{N}$, is satisfied by a function $f\not\equiv 0$ in $\mathcal{N}$, are the Blaschke sequences.

We mention that according to a classical theorem of Blaschke the zeros z_n of a nonidentically vanishing function in the class $\mathcal{N}$ form a Blaschke sequence [2, p.18; 1].

2. Two Auxiliary Lemmas

In order to prove our theorem, we state first two auxiliary lemmas.

LEMMA 1. *Let $f\in\mathcal{N}$ and $f(z)\neq 0$ for $z\in D$. Then*

$$(1-|z|)\log|f(z)|>-M \qquad \text{for all} \quad z\in D,$$

where M is a positive constant depending only on the function f.

Proof. From our assumption it follows [2, p. 25] that $\log|f(z)|$ has a representation as a Poisson–Stieltjes integral

$$\log|f(z)|=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{it}|^2}\,dv(t), \qquad z\in D,$$

where $v(t)$ is a function of bounded variation on $[0, 2\pi]$.

Further it is known that there exist bounded nondecreasing functions $\mu_1(t)$ and $\mu_2(t)$, such that $v(t)=\mu_1(t)-\mu_2(t)$ for $t\in[0, 2\pi]$.

From this we obtain

$$\log|f(z)|=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{it}|^2}\,d\mu_1(t)-\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{it}|^2}\,d\mu_2(t)$$

$$\geqslant -\frac{1}{\pi}\frac{1}{1-|z|}\int_0^{2\pi}d\mu_2(t), \qquad z\in D,$$

which proves our assertion with $M=1/\pi\int_0^{2\pi}d\mu_2(t)$ if this integral is positive and M equal to an arbitrary positive constant if $\int_0^{2\pi}d\mu_2(t)=0$.

LEMMA 2. *Suppose that* z_n, $n\in\mathbb{N}$, *are distinct points in* D *with* $|z_n|\to 1$ *as* $n\to\infty$, *and* $\sum_{n=1}^{\infty}(1-|z_n|)=\infty$, *and suppose that* ε_n, $n\in\mathbb{N}$, *are positive numbers with* $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. If

$$\delta_n=\min\{\varepsilon_n, \tfrac{1}{2}\inf_{\substack{i\in\mathbb{N}\\ i\neq n}}|z_n-z_i|\}, \quad n\in\mathbb{N},$$

then for every Blaschke product $B(z)$ *the estimate*

$$|B(z_n)|\geqslant\exp\left(-\frac{1}{\delta_n^2}\right)$$

holds for infinitely many indices n.

Proof. First we show that for every Blaschke sequence $\{w_k\}$ there is a subsequence $\{z_{n_\mu}\}$ of $\{z_n\}$, depending on $\{w_k\}$, so that for every $\mu\in\mathbb{N}$ we have

$$\inf_{k\in\mathbb{N}}|z_{n_\mu}-w_k|\geqslant\delta_{n_\mu}. \tag{1}$$

If this is not true, then for all except at most finitely many z_n, there exists a $w\in\{w_k\}$, so that

$$|z_n-w|<\varepsilon_n, \tag{2}$$

and

$$|z_n-w|<\tfrac{1}{2}\inf_{\substack{i\in\mathbb{N}\\ i\neq n}}|z_n-z_i|\leqslant\tfrac{1}{2}|z_n-z_m| \qquad \text{for every} \quad m\neq n. \tag{3}$$

Each w corresponds exactly to one z_n, because otherwise we would have for the same w and for a z_m, $z_m \neq z_n$, the inequality

$$|z_m - w| < \tfrac{1}{2} \inf_{\substack{i \in \mathbb{N} \\ i \neq m}} |z_m - z_i| \leqslant \tfrac{1}{2} |z_n - z_m|,$$

which, together with (3), implies the impossible relation

$$|z_n - z_m| < |z_n - z_m|.$$

Of course it is quite possible that to one z_n there exist more than one $w \in \{w_k\}$, so that (2) and (3) hold.

In all cases we can assign to almost every z_n a w as above, denoted by w_{k_n}. Clearly these w_{k_n} are distinct and they form a subsequence of $\{w_k\}$.

Form (2) it follows that

$$1 - |z_n| < 1 - |w_{k_n}| + \varepsilon_n,$$

and consequently

$$\sum (1 - |z_n|) < \sum (1 - |w_{k_n}|) + \sum \varepsilon_n$$
$$< \sum_{k=1}^{\infty} (1 - |w_k|) + \sum_{n=1}^{\infty} \varepsilon_n < \infty,$$

where the summation in $\sum (1 - |z_n|)$, $\sum (1 - |w_{k_n}|)$ and $\sum \varepsilon_n$ extends over all n except finitely many.

So we deduce finally that $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$, in contradiction to our assumption about $\{z_n\}$, and this ensures the existence of a sequence as in (1).

Let us now consider an arbitrary Blaschke product $B(z) = B(z, \{w_k\})$ in D with

$$|B(z_n)| < \exp\left(-\frac{1}{\delta_n^2}\right)$$

for all but a finite number of n's.

We observe that for every $z \notin \{w_k\}$,

$$\log \frac{1}{|B(z)|} = \sum_{k=1}^{\infty} \frac{1}{2} \log \left| \frac{1 - \bar{w}_k z}{z - w_k} \right|^2$$
$$= \sum_{k=1}^{\infty} \frac{1}{2} \log \left(\frac{(1 - |z|^2)(1 - |w_k|^2)}{|z - w_k|^2} + 1 \right)$$
$$< (1 - |z|^2) \sum_{k=1}^{\infty} \frac{1 - |w_k|}{|z - w_k|^2}$$

(see also [4, p. 508]).

Hence we get for almost all z_{n_μ}, which satisfy (2), the inequality

$$\delta_{n_\mu}^{-2} < \log \frac{1}{|B(z_{n_\mu})|} < (1 - |z_{n_\mu}|^2)\, \delta_{n_\mu}^{-2} \sum_{k=1}^{\infty} (1 - |w_k|).$$

Since $|z_{n_\mu}| \to 1$ as $\mu \to \infty$, this implies that $\sum_{k=1}^{\infty}(1 - |w_k|) = \infty$, in contradiction to the fact that $\{w_k\}$ is a Blaschke sequence.

3. Proof of the Theorem

First we prove that if the assumption of the theorem holds for a given sequence $\{z_n\}$ of distinct points in D, then $|z_n| \to 1$ as in $n \to \infty$.

Otherwise there is a subsequence $\{z_{n_\lambda}\}$ of $\{z_n\}$, with $z_{n_\lambda} \to z_0 \in D$ as $\lambda \to \infty$. Form $a_n^* < \exp[-1/\delta_n^2]$, $n \in \mathbb{N}$, in combination with the fact that $\delta_n \to 0$ as $n \to \infty$, it follows that $f(z_0) = 0$.

This implies for a $\gamma > 0$, an $m \in \mathbb{N}$, and for λ large enough,

$$\gamma\, |z_{n_\lambda} - z_0|^m < |f(z_{n_\lambda})| < \exp\left(-\frac{1}{\delta_{n_\lambda}^2}\right) < \exp\left(-\frac{1}{|z_{n_\lambda} - z_0|^2}\right),$$

which is impossible, because

$$|z_{n_\lambda} - z_0|^{-m} \exp\left(-\frac{1}{|z_{n_\lambda} - z_0|^2}\right) \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

Let now $\{z_n\}$ be a given sequence of distinct points in D, with $|z_n| \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty} (1 - |z_n|) = \infty$. Assuming that there is a function $f \not\equiv 0$ in $\mathcal{N}$ with

$$|f(z_n)| < \exp\left(-\frac{\lambda_n}{1 - |z_n|} - \frac{1}{\delta_n^2}\right) \qquad \text{for all} \quad n \in \mathbb{N},$$

and using Lemmas 1 and 2 we obtain a contradiction.

It is known that the function f has a factorization of the form

$$f(z) = \Phi(z)\, B(z), \quad z \in D,$$

where $B(z)$ is a Blaschke product and $\Phi \in \mathcal{N}$ with $\Phi(z) \neq 0$, $z \in D$ [2, p. 25].

In view of Lemma 2 there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$, such that

$$|B(z_{n_k})| \geqslant \exp\left(-\frac{1}{\delta_{n_k}^2}\right) \qquad \text{for all} \quad k \in \mathbb{N},$$

with δ_n defined as in the formulation of Lemma 2.

From this we deduce

$$|\Phi(z_{n_k})| = \frac{|f(z_{n_k})|}{|B(z_{n_k})|} < \exp\left(-\frac{\lambda_{n_k}}{1-|z_{n_k}|}\right),$$

or

$$(1-|z_{n_k}|)\log|\Phi(z_{n_k})| < -\lambda_{n_k} \qquad \text{for all } k.$$

The last inequality implies that

$$(1-|z_{n_k}|)\log|\Phi(z_{n_k})| \to -\infty \qquad \text{as} \quad k\to\infty,$$

which is impossible by Lemma 1. This completes the proof of the theorem.

4. Remarks

1. The condition that the z_n should be distinct cannot be ommited in our theorem. This can be shown by the following example.

We consider the sequence $\{r_k\}$, $r_k = 1-1/k^2$, $k\in\mathbb{N}$, and then the sequence $\{\rho_n\} = \{r_1, r_2, r_2, r_3, r_3, r_3, ...\}$ which consists of the points r_k, $k\in\mathbb{N}$, each taken k times.

It holds

$$\sum_{n=1}^{\infty}(1-\rho_n) = \sum_{k=1}^{\infty} k(1-r_k) = \sum_{k=1}^{\infty}\frac{1}{k} = \infty,$$

even though for the Blaschke product $B(z) = B(z, \{r_k\})$ and for every sequence $\{a_n\}$, $a_n > 0$, $n\in\mathbb{N}$, we have

$$|B(r_n)| = 0 < a_n \qquad \text{for all } n.$$

2. If the function f in the statement of our theorem is analytic in D but not in the Nevanlinna class, then our proposition is in general false.

A counterexample is given by the function

$$f(z) = \prod_{n=2}^{\infty}\left(1-\left(\frac{n}{n-1}z\right)^n\right).$$

This function is analytic in D and has n distinct zeros on the circle $|z| = 1-1/n$ for every positive integer $n\geqslant 2$, so it vanishes on a sequence $\{z_n\}$ which is not Blaschke (see also [3]).

3. In the statement of our theorem we give a critical sequence $\{a_n^*\}$, which depends on n and on the position of the z_n. The following example shows that it is not possible to replace $\{a_n^*\}$ by a sequence $\{\tilde{a}_n\}$, $\tilde{a}_n > 0$, $n \in \mathbb{N}$, $\tilde{a}_n \to 0$, depending on n alone.

We consider a sequence $\{z_n\}$ of distinct points in D, with

$$|z_n| = r_k = 1 - \frac{1}{k^2}$$

and

$$|z_n - r_k| < \frac{1}{k^2} \min\{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_{k(k+1)/2}\},$$

for every $k \in \mathbb{N}$ and every $n = k(k-1)/2 + 1, ..., k(k+1)/2$.

Let now n be arbitrary in $\mathbb{N}$ and let k_n be the uniquely determined natural number with $n \in [k_n(k_n - 1)/2 + 1, k_n(k_n + 1)/2]$.

For the Blaschke product $B(z) = B(z, \{r_k\})$ we have

$$|B(z_n)| < \left|\frac{z_n - r_{k_n}}{1 - r_{k_n} z_n}\right| < \frac{|z_n - r_{k_n}|}{1 - r_{k_n}} < \tilde{a}_n,$$

although $\sum_{n=1}^{\infty} (1 - |z_n|) = \sum_{k=1}^{\infty} 1/k = \infty$.

However, it remains an open quenstion if we can replace $\{a_n^*\}$ by a critical sequence depending on the position of the z_n alone.

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